

HERMITIAN AZUMAYA MODULES AND ARITHMETIC CHERN CLASSES

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ABSTRACT. We compute arithmetic Chern classes of sheaves on an arithmetic surface X associated to a Hermitian Azumaya algebra.

INTRODUCTION

Let $\pi : X \rightarrow Y$ be an arithmetic surface and let \mathcal{A} be an Azumaya algebra on X . In [Ree15] we introduced the notion of Hermitian Azumaya algebras and Hermitian Azumaya modules by equipping certain sheaves with Hermitian metrics. Using this, we defined a Deligne pairing $\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}$ for Hermitian \mathcal{A} -line bundles $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$, which generalized the classical Deligne pairing for line bundles on X .

In this note we introduce arithmetic Chern classes for Hermitian Azumaya modules and compute the first arithmetic Chern class of the Deligne pairing for \mathcal{A} -line bundles.

Our main result shows, that we can compute the first arithmetic Chern class of the \mathcal{A} -Deligne pairing in terms of the arithmetic \mathcal{A} -Chern classes. Explicitly it says:

$$\hat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = -\pi_*(\hat{c}_1^{\mathcal{A}}(\overline{\mathcal{M}})\hat{c}_1^{\mathcal{A}}(\overline{\mathcal{N}})).$$

The structure of this paper is as follows: In section 1 we recall some facts about Hermitian vector space, the construction of Hermitian inner products on associated vector spaces and isometric vector spaces. In section 2 we recall the definition of Hermitian Azumaya algebras and compute the first arithmetic Chern class of a Hermitian Azumaya algebra. In the final section 3 we introduce arithmetic Chern classes for Hermitian Azumaya modules and proof the main result.

1. HERMITIAN VECTOR SPACES

Definition 1.1. A Hermitian vector space \overline{V} is a pair (V, h) , where V is a finite dimensional \mathbb{C} -vector space and $h : V \times V \rightarrow \mathbb{C}$ is a Hermitian inner product.

The inner product h induces the so called Riesz isomorphism between V and the dual space $V^{\vee} = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ defined by

$$\theta_V : V \rightarrow V^{\vee}, v \mapsto h(-, v).$$

Note that this is a conjugate linear isomorphism. Using this isomorphism we define the associated dual hermitian vector space $\overline{V^{\vee}}$ to be the pair (V^{\vee}, h^{\vee}) , where h^{\vee} is the induced dual inner product defined by:

$$h^{\vee}(f, f') := \overline{h(\theta_V^{-1}(f), \theta_V^{-1}(f'))} \text{ for } f, f' \in V^{\vee}$$

This inner product gives the Riesz isomorphism $\theta_{V^{\vee}} : V^{\vee} \rightarrow V^{\vee\vee}$.

Iterating this construction we get the induced bidual Hermitian vector space $\overline{V^{\vee\vee}} = (V^{\vee\vee}, h^{\vee\vee})$.

Given two Hermitian vector spaces $\overline{V} = (V, h)$ and $\overline{W} = (W, k)$ we say $\Psi : V \rightarrow W$ induces an isometry of Hermitian vector spaces if Ψ is an isomorphism of \mathbb{C} -vector spaces

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and further more $\Psi^*k = h$, that is we have, for any two $v, v' \in V$ the equality $h(v, v') = k(\Psi(v), \Psi(v'))$.

Lemma 1.2. *The natural isomorphism $\iota : V \rightarrow V^{\vee\vee}$ induces an isometry of Hermitian vector spaces*

$$\iota : \overline{V} \xrightarrow{\sim} \overline{V^{\vee\vee}}.$$

Proof. We note that we have $\iota = \theta_{V^\vee} \circ \theta_V$. This can be seen as follows: for arbitrary $v \in V$ and $f \in V^\vee$ we have

$$\begin{aligned} ((\theta_{V^\vee} \circ \theta_V)(v))(f) &= h^\vee(f, \theta_V(v)) \\ &= \overline{h(\theta_V^{-1}(f), \theta_V^{-1}(\theta_V(v)))} \\ &= h(v, \theta_V^{-1}(f)) \\ &= f(v) = \iota(v)(f). \end{aligned}$$

Using this fact we can compute that we have $\iota^*h^{\vee\vee} = h$, hence ι is an isometry between \overline{V} and $\overline{V^{\vee\vee}}$. \square

Equip the tensor product $V \otimes W$ with the induced tensor product metric $h \otimes k$ defined on elementary tensors by:

$$h \otimes k(v \otimes w, v' \otimes w') := h(v, v')k(w, w').$$

This construction defines the Hermitian vector space $\overline{V \otimes W}$.

Remark 1.3. The Riesz isomorphism $\theta_{V \otimes W}$ induced by $h \otimes k$ has the following decomposition:

$$\theta_{V \otimes W} = \alpha \circ (\theta_V \otimes \theta_W).$$

Here $\alpha : V^\vee \otimes W^\vee \rightarrow (V \otimes W)^\vee$ is the natural isomorphism defined on elementary tensors by

$$f \otimes g \mapsto (v \otimes w \mapsto f(v)g(w)).$$

Lemma 1.4. *The natural isomorphism $\alpha : V^\vee \otimes W^\vee \rightarrow (V \otimes W)^\vee$ induces an isometry of Hermitian vector spaces*

$$\alpha : \overline{V^\vee \otimes W^\vee} \xrightarrow{\sim} \overline{(V \otimes W)^\vee}.$$

Proof. We have to show that $\alpha^*(h \otimes k)^\vee = h^\vee \otimes k^\vee$. Using the aforementioned decomposition of $\theta_{V \otimes W}$ we have:

$$\begin{aligned} \alpha^*(h \otimes k)^\vee(f \otimes g, f' \otimes g') &= \overline{h \otimes k(\theta_{V \otimes W}^{-1}(\alpha(f \otimes g)), \theta_{V \otimes W}^{-1}(\alpha(f' \otimes g')))} \\ &= \overline{h \otimes k(\theta_V^{-1} \otimes \theta_W^{-1}(f \otimes g), \theta_V^{-1} \otimes \theta_W^{-1}(f' \otimes g'))} \\ &= \overline{h(\theta_V^{-1}(f), \theta_V^{-1}(f'))} \overline{k(\theta_W^{-1}(g), \theta_W^{-1}(g'))} \\ &= h^\vee(f, f') \quad k^\vee(g, g') \\ &= h^\vee \otimes k^\vee(f \otimes g, f' \otimes g'). \end{aligned}$$

\square

Remark 1.5. We also note that the natural isomorphism

$$V \otimes W \xrightarrow{\sim} W \otimes V$$

induces an isometry of Hermitian vector spaces.

Composing all these natural isomorphisms we get a natural isomorphism

$$(1) \quad V^\vee \otimes W \cong V^\vee \otimes W^{\vee\vee} \cong W^{\vee\vee} \otimes V^\vee \cong (W^\vee \otimes V)^\vee$$

which, as we showed, induces in fact an isometry of Hermitian vector spaces:

$$\overline{V^\vee \otimes W} \xrightarrow{\sim} \overline{(W^\vee \otimes V)^\vee}.$$

Using the natural isomorphisms

$$V^\vee \otimes W \cong \operatorname{Hom}(V, W) \text{ and } W^\vee \otimes V \cong \operatorname{Hom}(W, V),$$

the isomorphism 1 gives rise to a natural isomorphism

$$\operatorname{Hom}(V, W) \xrightarrow{\sim} \operatorname{Hom}(W, V)^\vee.$$

It is well known that this isomorphism is nothing but the trace map, i.e. it comes from the perfect pairing:

$$\operatorname{Hom}(V, W) \times \operatorname{Hom}(W, V) \xrightarrow{(-) \circ (-)} \operatorname{Hom}(V, V) \xrightarrow{\operatorname{tr}} \mathbb{C}$$

and it maps $\phi : V \rightarrow W \in \operatorname{Hom}(V, W)$ to $\operatorname{tr}((-) \circ \phi) \in \operatorname{Hom}(W, V)^\vee$.

The natural isomorphism $V^\vee \otimes W \xrightarrow{\sim} \operatorname{Hom}(V, W)$ also defines a Hermitian inner product on $\operatorname{Hom}(V, W)$ which in turn defines the Hermitian vector space $\overline{\operatorname{Hom}(V, W)}$.

Putting all these steps together we have proven:

Theorem 1.6. *The trace map $\operatorname{tr} : \operatorname{Hom}(V, W) \rightarrow \operatorname{Hom}(W, V)^\vee$ induces an isometry of Hermitian vector spaces*

$$\operatorname{tr} : \overline{\operatorname{Hom}(V, W)} \xrightarrow{\sim} \overline{\operatorname{Hom}(W, V)^\vee}.$$

Remark 1.7. All results in this section can be immediately generalized to Hermitian vector bundles, that is pairs (E, h) , where E is a complex vector bundle on a smooth manifold and h is a Hermitian metric on E , that is a Hermitian inner product on each fiber.

2. HERMITIAN AZUMAYA ALGEBRAS

Definition 2.1. We call a two-dimensional integral regular projective flat \mathbb{Z} -scheme X an arithmetic surface and denote the structure morphism by $\pi : X \rightarrow Y$, here we define $Y := \operatorname{Spec}(\mathbb{Z})$.

We denote the generic fiber of π by $X_{\mathbb{Q}}$ and the base change to \mathbb{C} by $X_{\mathbb{C}}$. Both $X_{\mathbb{Q}}$ and $X_{\mathbb{C}}$ are smooth curves over \mathbb{Q} resp. \mathbb{C} .

Given a sheaf of \mathcal{O}_X -modules \mathcal{F} , then we denote the induced sheaves on $X_{\mathbb{Q}}$ and $X_{\mathbb{C}}$ by $\mathcal{F}_{\mathbb{Q}}$ and $\mathcal{F}_{\mathbb{C}}$.

The associated Riemann surface S of X is given by $S = X_{\mathbb{C}}(\mathbb{C})$. S comes endowed with a Hermitian metric ω , which is Kähler. If \mathcal{F} is a locally free \mathcal{O}_X -module, then we denote the induced vector bundle on S by F .

Definition 2.2. A Hermitian vector bundle $\overline{\mathcal{E}}$ on X is a pair (\mathcal{E}, h) , where \mathcal{E} is a locally free sheaf on X and h is a Hermitian metric on the induced vector bundle E on S , which is invariant under the complex conjugation on S .

Remark 2.3. If we use a Hermitian metric in the following, we will always assume that the metric is invariant with respect to the complex conjugation on S , see [Sou92, Definition IV.4.1.4.].

Let \mathcal{A} be an Azumaya algebra on the arithmetic surface X , that is a matrix algebra in the étale topology, then we have

$$(2) \quad \mathcal{A}_{\mathbb{C}} \cong \operatorname{End}_{\mathcal{O}_{X_{\mathbb{C}}}}(\mathcal{E})$$

for some locally free sheaf \mathcal{E} on $X_{\mathbb{C}}$. This fact is due to Tsen's theorem. The sheaf \mathcal{E} induces a vector bundle E on S , especially we have $\mathcal{A} \cong \operatorname{End}(\mathcal{E}) \cong E^\vee \otimes E$ on S .

Choosing a Hermitian metric h on E induces the metric $h^\vee \otimes h$ on $E^\vee \otimes E$, and hence also a metric $h^{\mathcal{A}}$ on \mathcal{A} . This leads to the following definition:

Definition 2.4. Let X be an arithmetic surface. A Hermitian Azumaya algebra $\overline{\mathcal{A}}$ over X is a pair $(\mathcal{A}, h^{\mathcal{A}})$, here \mathcal{A} is Azumaya algebra on X and $h^{\mathcal{A}}$ a Hermitian metric on the associated vector bundle constructed as described above.

Theorem 2.5. *Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X , then we have:*

$$2\widehat{c}_1(\overline{\mathcal{A}}) = 0 \in \widehat{CH}^1(X).$$

Proof. The trace pairing for the algebra \mathcal{A} on X , given by

$$\mathrm{tr} : \mathcal{A} \rightarrow \mathcal{A}^\vee, \quad a \mapsto (b \mapsto \mathrm{tr}(ba)),$$

is an isomorphism. This can be checked étale locally, where it reduces to the fact that \mathcal{A} becomes a matrix algebra and this algebra is self-dual with respect to the trace map.

Now the Hermitian Azumaya algebra $\overline{\mathcal{A}}$ induces a Hermitian vector bundle $\overline{\mathcal{A}}^\vee$ via the dual metric. Theorem 1.6 says that the trace induces an isometric isomorphism

$$\mathrm{tr} : A \cong \mathrm{End}(E) \xrightarrow{\sim} \mathrm{End}(E)^\vee \cong A^\vee.$$

by our choice of the metric on the Hermitian Azumaya algebra.

We especially have

$$\mathrm{tr}^*((h^{\mathcal{A}})^\vee) = h^{\mathcal{A}},$$

i.e. $h^{\mathcal{A}}$ is the metric induced by the trace map and $(h^{\mathcal{A}})^\vee$.

On the one hand we now compute

$$\widehat{c}_1(\overline{\mathcal{A}}^\vee) = \widehat{c}_1(\overline{\mathcal{A}}).$$

This follows from the above and [GS90, Proposition 1.2.5, Theorem 4.8.(ii)] since $h^{\mathcal{A}}$ is the metric induced by the metric on \mathcal{A}^\vee via the trace map.

On the other hand we have, using [GS90, 4.9.],

$$\widehat{c}_1(\overline{\mathcal{A}}^\vee) = -\widehat{c}_1(\overline{\mathcal{A}}).$$

Combining these results gives:

$$\widehat{c}_1(\overline{\mathcal{A}}) = \widehat{c}_1(\overline{\mathcal{A}}^\vee) = -\widehat{c}_1(\overline{\mathcal{A}}),$$

or equivalently

$$2\widehat{c}_1(\overline{\mathcal{A}}) = 0.$$

□

Corollary 2.6. *Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X , then $\widehat{c}_1(\overline{\mathcal{A}}) = 0$ if $\widehat{CH}^1(X)$ is torsion-free.*

Corollary 2.7. *Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X , then $\widehat{c}_1(\overline{\mathcal{A}}) = 0 \in \widehat{CH}^1(X)_\mathbb{Q}$, especially $\widehat{ch}(\overline{\mathcal{A}}) = \mathrm{rk}(\mathcal{A}) - \widehat{c}_2(\overline{\mathcal{A}}) \in \widehat{CH}(X)_\mathbb{Q}$.*

3. THE FIRST ARITHMETIC CHERN CLASS OF THE DELIGNE PAIRING

Let X be an arithmetic surface. If $\overline{\mathcal{A}}$ is a Hermitian Azumaya algebra on X and \mathcal{M} is a locally projective left \mathcal{A} -module, then using 2 and Morita equivalence, we see that

$$\mathcal{M}_\mathbb{C} = \mathcal{E} \otimes_{\mathcal{O}_{X_\mathbb{C}}} \mathcal{M}'$$

for some locally free sheaf \mathcal{M}' on $X_\mathbb{C}$ which induces a vector bundle M' on S , so the induced vector bundle M on S is given by $M = E \otimes M'$.

The vector bundle E still comes with the Hermitian metric h and we furthermore pick a Hermitian metric h' on M' . The tensor product metric of h and h' yields a Hermitian metric $h^{\mathcal{M}} := h \otimes h'$ on M . This defines a Hermitian locally free sheaf $\overline{\mathcal{M}} = (\mathcal{M}, h^{\mathcal{M}})$, which also has the structure of a locally projective left \mathcal{A} -module. This suggests the following definition:

Definition 3.1. Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra over X . A Hermitian Azumaya module $\overline{\mathcal{M}}$ is a couple $(\mathcal{M}, h^{\mathcal{M}})$ where \mathcal{M} is a locally projective \mathcal{A} -module and $h^{\mathcal{M}}$ is a Hermitian metric on the associated vector bundle on S chosen as described above.

Assume $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ are two Hermitian Azumaya modules, then $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is locally free as an \mathcal{O}_X -module as both modules are locally projective over \mathcal{A} . We remember that we have $\mathcal{A}_{\mathbb{C}} \cong \mathcal{E}nd_{\mathcal{O}_{X_{\mathbb{C}}}}(\mathcal{E})$ for some locally free sheaf \mathcal{E} on $X_{\mathbb{C}}$ and $\mathcal{M}_{\mathbb{C}} \cong \mathcal{E} \otimes \mathcal{M}'$ as well as $\mathcal{N}_{\mathbb{C}} \cong \mathcal{E} \otimes \mathcal{N}'$ by Morita equivalence. This equivalence also gives a natural isomorphism

$$\mathcal{H}om_{\mathcal{A}_{\mathbb{C}}}(\mathcal{M}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}) \cong \mathcal{H}om_{\mathcal{O}_{X_{\mathbb{C}}}}(\mathcal{M}', \mathcal{N}').$$

The vector bundle associated to the last sheaf is isomorphic to $(M')^{\vee} \otimes N'$.

This bundle comes naturally equipped with the Hermitian metric $(h')^{\vee} \otimes h''$, the one given by the Hermitian metrics h' and h'' on the vector bundles M' and N' . So we also have a Hermitian metric $h^{(\mathcal{M}, \mathcal{N})}$ on the vector bundle associated to $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$. This construction defines the Hermitian vector bundle

$$\overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})} := (\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N}), h^{(\mathcal{M}, \mathcal{N})}).$$

The trace map and the tensor-hom adjunction give the following natural isomorphism of locally free \mathcal{O}_X -modules:

$$\begin{aligned} \mathcal{A} \otimes \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) &\cong \mathcal{A}^{\vee} \otimes \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \\ &\cong \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{M}, \mathcal{N}) \\ &\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{N})) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}). \end{aligned}$$

Lemma 3.2. *The natural isomorphism $\mathcal{A} \otimes \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ induces an isometry of Hermitian vector bundles*

$$\overline{\mathcal{A} \otimes \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})} \xrightarrow{\sim} \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})}.$$

Proof. Looking at all the isomorphisms and the naturally induced metrics on the vector bundles on S , this boils down to the following isometry

$$(E^{\vee} \otimes E \otimes M'^{\vee} \otimes N', h^{\vee} \otimes h \otimes (h')^{\vee} \otimes h'') \cong ((E \otimes M')^{\vee} \otimes E \otimes N', (h \otimes h')^{\vee} \otimes h \otimes h'')$$

which follows from 1.4 and 1.5. \square

Corollary 3.3. *Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X , if $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ are Hermitian Azumaya modules, then we have the following equality:*

$$\widehat{ch}(\overline{\mathcal{A}}) \widehat{ch}(\overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})}) = \widehat{ch}(\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})})$$

Proof. By 3.2 the Hermitian vector bundles $\overline{\mathcal{A} \otimes \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})}$ and $\overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})}$ are isometric, so the properties of the arithmetic Chern classes give the desired result. \square

Definition 3.4. Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X , then we define the arithmetic \mathcal{A} -Chern character of a Hermitian Azumaya module $\overline{\mathcal{M}}$ by:

$$\widehat{ch}^{\mathcal{A}}(\overline{\mathcal{M}}) := \widehat{ch}(\overline{\mathcal{M}}) \widehat{ch}(\overline{\mathcal{A}})^{-\frac{1}{2}}$$

and the first arithmetic \mathcal{A} -Chern class by

$$\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{M}}) := (\widehat{ch}^{\mathcal{A}}(\overline{\mathcal{M}}))^{(1)}.$$

Remark 3.5. By the definition of the arithmetic \mathcal{A} -Chern character and by the choice of the induced metrics on all sheaves involved, we see:

$$\widehat{ch}(\overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})}) = \widehat{ch}^{\mathcal{A}}(\overline{\mathcal{M}}) \widehat{ch}^{\mathcal{A}}(\overline{\mathcal{N}}).$$

Explicitly we have

$$\widehat{ch}(\overline{\mathcal{A}})^{-\frac{1}{2}} = \frac{1}{\sqrt{rk(\overline{\mathcal{A}})}} - \frac{1}{2\sqrt{rk(\overline{\mathcal{A}})}^3} \widehat{c}_2(\overline{\mathcal{A}}).$$

This computation shows:

$$\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{M}}) = \frac{1}{\sqrt{rk(\overline{\mathcal{A}})}} \widehat{c}_1(\overline{\mathcal{M}}).$$

A similar explicit computation is possible for $\widehat{ch}^{\mathcal{A}}(\overline{\mathcal{M}})$.

Since \mathcal{A} is an Azumaya algebra, $rk(\mathcal{A})$ is a square, so all these classes are well defined in $\widehat{CH}(X)_{\mathbb{Q}}$.

Now we want to compute some arithmetic Chern classes, using the arithmetic Riemann-Roch theorem, due to Gillet and Soulé, see [GS92]. As X is an arithmetic surface, we have $\dim(X) - \dim(Y) = 1$, so we can use the following version of the arithmetic Riemann-Roch theorem for a Hermitian vector bundle $\overline{\mathcal{E}}$ on X , see [GS91, Conjecture 1.5]:

$$(3) \quad \widehat{c}_1(\overline{\lambda(\mathcal{E})}) = \pi_*(\widehat{ch}(\overline{\mathcal{E}})\widehat{Td}(\overline{\mathcal{T}_{X/Y}}))^{(1)} - a(rk(\mathcal{E})(1-g)(4\zeta'(-1) - \frac{1}{6}))$$

Here $\overline{\lambda(\mathcal{E})}$ is the 'determinant of the cohomology', a Hermitian line bundle on Y defined by

$$\lambda(\mathcal{E}) := \bigotimes_{i \geq 0} \det(R^i \pi_* \mathcal{E})^{(-1)^i},$$

and the line bundle $\lambda(\mathcal{E})$ is equipped with the Quillen metric h_Q induced by the Hermitian metric on \mathcal{E} . Furthermore g is the genus of $X_{\mathbb{Q}}$ and $\zeta'(-1)$ is the value of the derivative of the Riemann zeta function at -1 .

Definition 3.6. Let X be an arithmetic surface and $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X . If $(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ is a pair of Hermitian Azumaya modules, then we define the \mathcal{A} -Deligne pairing of the pair $\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}$ as the Hermitian line bundle on Y given by:

$$\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}} := \overline{\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N})} \otimes \overline{\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{A})}^{(-1)} \otimes \overline{\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{N})}^{(-1)} \otimes \overline{\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{A})},$$

where

$$\overline{\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N})} = \overline{\lambda(\text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}))}.$$

Theorem 3.7. Let X be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on X . If $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ are Hermitian \mathcal{A} -line bundles, that is $rk(\mathcal{A}) = rk(\mathcal{M}) = rk(\mathcal{N})$, then there is the following equality in $\widehat{CH}^1(X)_{\mathbb{Q}}$:

$$\widehat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = -\pi_*(\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{M}})\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{N}})).$$

Proof. By the properties of \widehat{c}_1 we have:

$$\widehat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = \widehat{c}_1(\overline{\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N})}) - \widehat{c}_1(\overline{\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{A})}) - \widehat{c}_1(\overline{\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{N})}) + \widehat{c}_1(\overline{\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{A})}).$$

Using the arithmetic Riemann-Roch theorem 3, 3.3 and 3.5 one gets:

$$\widehat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = \pi_*((\widehat{ch}(\overline{\mathcal{M}^{\vee}}) - \widehat{ch}(\overline{\mathcal{A}^{\vee}}))(\widehat{ch}(\overline{\mathcal{N}}) - \widehat{ch}(\overline{\mathcal{A}}))\widehat{ch}(\overline{\mathcal{A}})^{-1}\widehat{Td}(\overline{\mathcal{T}_{X/Y}}))^{(1)}$$

since the analytic terms of the form $a(-)$ cancel each other.

Using the definitions of \widehat{ch} one computes:

$$\widehat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = \pi_*(\widehat{c}_1(\overline{\mathcal{M}^{\vee}})\widehat{c}_1(\overline{\mathcal{N}})\widehat{ch}(\overline{\mathcal{A}})^{-1}\widehat{Td}(\overline{\mathcal{T}_{X/Y}}))^{(1)}$$

since $rk(\mathcal{A}) = rk(\mathcal{M}) = rk(\mathcal{N})$.

So using the definition of $\widehat{c}_1^{\mathcal{A}}$ and \widehat{Td} we finally get:

$$\widehat{c}_1(\langle \overline{\mathcal{M}}, \overline{\mathcal{N}} \rangle_{\mathcal{A}}) = -\pi_*(\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{M}})\widehat{c}_1^{\mathcal{A}}(\overline{\mathcal{N}})).$$

□

Remark 3.8. The sign in the formula is not surprising, since the Deligne pairing for Hermitian Azumaya modules is the dual of the classical Deligne pairing in the case $\overline{\mathcal{A}} = \overline{\mathcal{O}_X}$, see [Ree15, Theorem 2.7.].

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